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Thermofield Dynamics of the Heterotic String
— Physical Aspects of the Thermal Duality —

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ABSTRACT

The thermofield dynamics of the $D = 10$ heterotic thermal string theory is described in proper reference to the thermal duality symmetry as well as the thermal stability of modular invariance in association with the global phase structure of the heterotic thermal string ensemble.

Building up thermal string theories based upon the thermofield dynamics (TFD) [1] has gradually been endeavoured in leaps and bounds [2] – [9]. In the present communication, physical aspects of the thermal duality symmetry [10] – [13] are commentarially exemplified *à la* recent publication of ourselves [8] through the infrared behaviour of the one-loop cosmological constant in proper respect of the thermal stability of modular invariance for the $D = 10$ heterotic thermal string theory based upon the TFD algorithm. The global phase structure of the $D = 10$ heterotic thermal string ensemble is then recapitulatively touched upon.

Let us start with the one-loop cosmological constant $\Lambda(\beta)$ as follows:

$$\Lambda(\beta) = \frac{\alpha'}{2} \lim_{\mu^2 \to 0} \mathrm{Tr} \left[\int_{\infty}^{\mu^2} dm^2 \left(\Delta_B^\beta(p, P; m^2) + \Delta_F^\beta(p, P; m^2) \right) \right] \tag{1}$$

at any finite temperature $\beta^{-1} = kT$ in the $D = 10$ heterotic thermal string theory based upon the TFD algorithm, where α' means the slope parameter, p^μ reads loop momentum, P^I lie on the even self-dual root lattice $L = \Gamma_8 \times \Gamma_8$ for the exceptional group $G = E_8 \times E_8$ [14] and the thermal propagator $\Delta_{B[F]}^\beta(p, P; m^2)$ of the free closed bosonic [fermionic] string is written *à la* Leblanc [2] in the form

$$\begin{aligned}\Delta_{B[F]}^\beta(p, P; m^2) &= \int_{-\pi}^{\pi} \frac{d\phi}{4\pi}\, \mathrm{e}^{i\phi\left(N - \alpha - \bar{N} + \bar{\alpha} - 1/2 \cdot \sum_{I=1}^{16} (P^I)^2\right)} \\ &\times \left(\left[\underset{[-]}{+} \int_0^1 dx + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\delta[\alpha'/2 \cdot p^2 + \alpha'/2 \cdot m^2 + 2(n - \alpha)]}{\mathrm{e}^{\beta|p_0|} \underset{[+]}{-} 1} \oint_c dx \right] \right. \\ &\left. \times x^{\alpha'/2 \cdot p^2 + N - \alpha + \bar{N} - \bar{\alpha} + 1/2 \cdot \sum_{I=1}^{16} (P^I)^2 + \alpha'/2 \cdot m^2 - 1} \right) \quad ,\end{aligned} \tag{2}$$

where N $[\bar{N}]$ denotes the number operator of the right- [left-] mover, the intercept parameter α $[\bar{\alpha}]$ is fixed at $\alpha = 0$ $[\bar{\alpha} = 1]$ and the contour c is taken as the unit circle around the origin. We are then eventually led to the modular parameter integral representation of $\Lambda(\beta)$ at $D = 10$ as follows [8]:

$$
\begin{aligned}
\Lambda(\beta) &= -8(2\pi\alpha')^{-D/2}\int_E \frac{d^2\tau}{2\pi\tau_2^2}\,(2\pi\tau_2)^{-(D-2)/2}\,\mathrm{e}^{2\pi i\bar\tau}\left[1+480\sum_{m=1}^{\infty}\sigma_7(m)\bar z^m\right] \\
&\quad\times\prod_{n=1}^{\infty}(1-\bar z^n)^{-D-14}\left(\frac{1+z^n}{1-z^n}\right)^{D-2}\sum_{\ell\in Z;\mathrm{odd}}\exp\left[-\frac{\beta^2}{4\pi\alpha'\tau_2}\,\ell^2\right] \quad ,
\end{aligned}
\tag{3}
$$

where $\overset{[-]}{\tau} = \tau_1 \underset{[-]}{+} i\tau_2$, $z = x\mathrm{e}^{i\phi} = \mathrm{e}^{2\pi i\tau}$, $\bar z = x\mathrm{e}^{-i\phi} = \mathrm{e}^{-2\pi i\bar\tau}$, E means the half-strip integration region in the complex τ plane, *i.e.* $-1/2 \le \tau_1 \le 1/2$; $\tau_2 > 0$. Accordingly, the $D = 10$ thermal amplitude $\beta\Lambda(\beta)$ is identical with the "E-type" representation of the thermo-partition function $\Omega_h(\beta)$ of the heterotic string in ref. [10]. The "E-type" thermal amplitude $\Lambda(\beta)$ is not modular invariant and annoyed with ultraviolet divergences for $\beta \le \beta_H = (2+\sqrt{2})\pi\sqrt{\alpha'}$, where β_H reads the inverse Hagedorn temperature of the heterotic thermal string. Let us pay attention to the fact that the thermal amplitude $\Lambda(\beta)$ is infrared convergent for any value of β.

Our prime concern is reduced to regularizing the thermal amplitude $\Lambda(\beta)$ *à la* ref. [10] as well as ref. [13] through transforming the physical information in the ultraviolet region of the half-strip E into the "new-fashioned" modular invariant amplitude. Let us postulate *à la* ref. [8] the one-loop dual symmetric thermal cosmological constant $\bar\Lambda(\beta; D)$ at any space-time dimension D as an integral over the fundamental domain F, *i.e.* $-1/2 \le \tau_1 \le 1/2\,;\tau_2 > 0\,;|\tau| > 1$, of the modular group $SL(2,Z)$ as follows:

$$
\begin{aligned}
\bar\Lambda(\beta; D) &= \frac{2}{\beta}(2\pi\alpha')^{-D/2}\sum_{(\sigma,\rho)}\int_F \frac{d^2\tau}{2\pi\tau_2^2}\,(2\pi\tau_2)^{-(D-2)/2}\,\bar z^{-(D+14)/24} z^{-(D-2)/24} \\
&\quad\times\left[1+480\sum_{m=1}^{\infty}\sigma_7(m)\bar z^m\right]\prod_{n=1}^{\infty}(1-\bar z^n)^{-D-14}(1-z^n)^{-D+2} \\
&\quad\times A_{\sigma\rho}(\tau; D)\left[C_\sigma^{(+)}(\bar\tau,\tau;\beta)+\rho C_\sigma^{(-)}(\bar\tau,\tau;\beta)\right] \quad ,
\end{aligned}
\tag{4}
$$

where

$$\begin{pmatrix} A_{+-}(\tau; D) \\ A_{-+}(\tau; D) \\ A_{--}(\tau; D) \end{pmatrix} = 8\left(\frac{\pi}{4}\right)^{(D-2)/6} \begin{pmatrix} -[\theta_2(0,\tau)/\theta_1'(0,\tau)^{1/3}]^{(D-2)/2} \\ -[\theta_4(0,\tau)/\theta_1'(0,\tau)^{1/3}]^{(D-2)/2} \\ [\theta_3(0,\tau)/\theta_1'(0,\tau)^{1/3}]^{(D-2)/2} \end{pmatrix} , \qquad (5)$$

$$C_\sigma^{(\gamma)}(\bar{\tau}, \tau; \beta) = (4\pi^2\alpha'\tau_2)^{1/2} \sum_{(p,q)} \exp\left[-\frac{\pi}{2}\left(\frac{\beta^2}{2\pi^2\alpha'}p^2 + \frac{2\pi^2\alpha'}{\beta^2}q^2\right)\tau_2 + i\pi pq\tau_1\right], \qquad (6)$$

the signatures σ, ρ and γ read $\sigma, \rho = +,-;\ -,+;\ -,-$ and $\gamma = +,-$, respectively, and the summation over p $[q]$ is restricted by $(-1)^p = \sigma$ $[(-1)^q = \gamma]$. It is almost needless to mention that the $D = 10$ thermal amplitude $\beta\bar{\Lambda}(\beta; D = 10)$ is literally reduced to the "D-type" representation of the thermo-partition function $\Omega_h(\beta)$ in ref. [10] which in turn guarantees $\bar{\Lambda}(\beta; D = 10) = \Lambda(\beta)$ at least below the Hagedorn temperature β_H^{-1}. Typical theoretical observations are as follows [8]: First of all, the thermal amplitude $\bar{\Lambda}(\beta; D)$ is manifestly modular invariant and free of ultraviolet divergences for any value of β and D. If and only if $D = 10$, in addition, the thermal duality relation $\beta\bar{\Lambda}(\beta; D) = \tilde{\beta}\bar{\Lambda}(\tilde{\beta}; D)$ is manifestly satisfied for the thermal amplitude $\bar{\Lambda}(\beta; D)$, irrespective of the value of β, where $\tilde{\beta} = 2\pi^2\alpha'/\beta$. By way of parenthesis, let us remember that the so-called thermal duality relation is not universal for closed string theories in general [10] – [13] in sharp contrast to modular invariance.

The infrared behaviour of the thermal cosmological constant $\bar{\Lambda}(\beta; D)$ is asymptotically described as [8]

$$\begin{aligned} \bar{\Lambda}(\beta; D) \quad &= \quad -64\sqrt{2}\,(8\pi^2\alpha')^{-D/2} \sum_{(p,q)} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp[i\pi pq\tau_1] \sqrt{\frac{\tilde{\beta}}{\beta}} \int_{\sqrt{1-\tau_1^2}}^{\infty} d\tau_2\ \tau_2^{-(D+1)/2} \\ &\quad \times \exp\left[-\frac{\pi}{2}\,\tau_2\left(\frac{\beta}{\tilde{\beta}}\,p^2 + \frac{\tilde{\beta}}{\beta}\,q^2 - \frac{5}{12}(D-10) - 6\right)\right] \quad , \end{aligned} \qquad (7)$$

where $p, q = \pm1; \pm3; \pm5; \cdots$. The $D = 10$ TFD amplitude $\bar{\Lambda}(\beta; D = 10)$ is then infrared divergent for $(2-\sqrt{2})\pi\sqrt{\alpha'} = \tilde{\beta}_H \leq \beta \leq \beta_H$ in association with the presence of the

tachyonic mode, where $\tilde{\beta}_H$ reads the inverse dual Hagedorn temperature of the heterotic thermal string. We can therefore define *à la* ref. [8] the dimensionally regularized, $D = 10$ one-loop dual symmetric thermal cosmological constant $\hat{\Lambda}(\beta)$ in the sense of analytic continuation from $D < 2/5$ to higher values of D, *i.e.* $D = 10$ by

$$\begin{aligned}\hat{\Lambda}(\beta) \;\; &= \;\; -\frac{2}{\beta}(8\pi\alpha')^{-(D-1)/2}\sum_{(p,q)}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp[i\pi pq\tau_1] \\ &\quad\times\left(\frac{\beta^2}{2\pi^2\alpha'}\,p^2+\frac{2\pi^2\alpha'}{\beta^2}\,q^2-6-i\varepsilon\right)^{(D-1)/2} \\ &\quad\times\Gamma\left[-\frac{D-1}{2}\,,\;\frac{\pi}{2}\sqrt{1-\tau_1^2}\left(\frac{\beta^2}{2\pi^2\alpha'}\,p^2+\frac{2\pi^2\alpha'}{\beta^2}\,q^2-6-i\varepsilon\right)\right]\;;\quad D=10,\end{aligned} \tag{8}$$

irrespective of the value of β, where Γ is the incomplete gamma function of the second kind and the so-called $D + i\varepsilon$ procedure is to be adopted in natural consonance with the nonvanishing decay rate of the tachyonic thermal vacuum.

The dimensionally regularized, thermal cosmological constant $\hat{\Lambda}(\beta)$ manifestly satisfies the thermal duality relation $\beta\hat{\Lambda}(\beta) = \tilde{\beta}\hat{\Lambda}(\tilde{\beta})$ in full accordance with the thermal stability of modular invariance. The thermal duality symmetry mentioned above immediately yields the asymptotic formula as follows [5], [11]:

$$\lim_{\beta\to 0}\beta\hat{\Lambda}(\beta) = \lim_{\tilde{\beta}^{-1}\to 0}\tilde{\beta}\hat{\Lambda}(\tilde{\beta})\;, \tag{9}$$

or equivalently

$$\hat{\Lambda}(\beta\sim 0)\sim\frac{2\pi^2\alpha'}{\beta^2}\hat{\Lambda}(\beta^{-1}\to 0)=\frac{2\pi^2\alpha'}{\beta^2}\Lambda \tag{10}$$

for the $D = 10$ heterotic thermal string theory, where Λ literally reads the $D = 10$ zero-temperature, one-loop cosmological constant which is in turn guaranteed to vanish automatically as an inevitable consequence of the Jacobi identity $\theta_2^4 - \theta_3^4 + \theta_4^4 = 0$ for the theta functions. Let us call to our remembrance that the vanishing machinery of

the $D = 10$ zero-temperature amplitude Λ is self-evident in the present context due to the absence of the term with $\ell = 0$ [$p = q = 0$] on the right hand side of eq. (3) [eq. (7) or equivalently eq. (8)]. The present observation is paraphrased *à la* ref. [13] as follows: The thermal duality symmetry is inherent to the fact that the total number of degrees of freedom vanishes at extremely high temperature $\beta \sim 0$ in the sense of the modular invariant counting. Accordingly, it seems possible to claim *à la* ref. [11] that the $D = 10$ heterotic thermal string has no fundamental gauge invariant degrees of freedom at least at $\beta \sim 0$ and will be asymptotically described at high temperature by underlying topological theory. The present view may be in essential agreement with the provocative argument of Witten [15] on the possible unbroken high-energy phase of string theory in topological σ models and will deserve more than passing consideration in an attempt to substantialize the crucial geometrical ideas purely topological in character, *e.g.* the possible background independence at asymptotically high energies in string theory. It is parenthetically mentioned that another newfangled hypothetical view might not be exhaustively excluded yet at least as a matter of taste in which $\bar{\Lambda}(\beta; D)$ and consequently $\hat{\Lambda}(\beta)$ would not be physically well-defined beyond the Hagedorn temperature β_H^{-1}.

Let us describe *à la* ref. [8] the singularity structure of the dimensionally regularized, dual symmetric thermal amplitude $\hat{\Lambda}(\beta)$. The position of the singularity $\beta_{|p|,|q|}$ is determined by solving $\beta/\tilde{\beta}\cdot p^2 + \tilde{\beta}/\beta\cdot q^2 - 6 = 0$ for every allowed (p, q) in eq. (8). Thus we obtain a set of solutions as follows: $\beta_{1,1} = \beta_H = (\sqrt{2}+1)\pi\sqrt{2\alpha'}$ and $\tilde{\beta}_{1,1} = \tilde{\beta}_H = (\sqrt{2}-1)\pi\sqrt{2\alpha'}$ which form the leading branch points of the square root type at β_H and $\tilde{\beta}_H$, respectively. It is of practical significance to note that there exists no self-dual leading branch point at $\beta_0 = \tilde{\beta}_0 = \pi\sqrt{2\alpha'}$. As a matter of fact, the regularized thermal amplitude $\hat{\Lambda}(\beta)$ develops the imaginary part across the leading branch cut $\tilde{\beta}_H \leq \beta \leq \beta_H$ in association with instability of the tachyonic thermal vacuum. We are now in the position to touch upon *à la* ref. [8] the global phase structure of the $D = 10$ heterotic thermal string ensemble. There will then exist three phases in the sense of the thermal duality symmetry

as follows [8], [10], [16]: (i) the β channel canonical phase in the tachyon-free region $(2+\sqrt{2})\pi\sqrt{\alpha'} = \beta_H \leq \beta < \infty$, (ii) the dual $\tilde{\beta}$ channel canonical phase in the tachyon-free region $0 < \beta \leq \tilde{\beta}_H = (2-\sqrt{2})\pi\sqrt{\alpha'}$ and (iii) the self-dual microcanonical phase in the tachyonic region $\tilde{\beta}_H < \beta < \beta_H$. There will appear no effective splitting of the microcanonical domain because of the absence of the self-dual branch point at $\beta_0 = \tilde{\beta}_0 = \pi\sqrt{2\alpha'}$. As a consequence, it still remains to be clarified whether the so-called maximum temperature of the heterotic string excitation is asymptotically described as $\beta_0^{-1} = \tilde{\beta}_0^{-1}$ in proper respect of the self-duality of the microcanonical phase. If the thermal duality relation were tentatively supposed to be manifestly broken, by way of parenthesis, there would then appear the essential singularity at the infinite temperature $\beta = 0$ as the accumulation point of infinitely many finite-temperature branch points beyond the Hagedorn temperature β_H^{-1} [3], [5], [7].

We have succeeded in shedding some light upon the global phase structure of the thermal string ensemble through the infrared behaviour of the one-loop free energy amplitude for the dimensionally regularized, $D = 10$ heterotic thermal string theory based upon the TFD algorithm. In particular, physical aspects of the thermal duality symmetry have been described in full harmony with the thermal stability of modular invariance not only for the canonical region but also for the microcanonical region. It is hoped that we can illuminate the fruitful thermodynamical investigation [17] of string excitations, *e.g.* the manifest materialization of the "true" maximum temperature for the thermal string ensemble in general within the new-fashioned duality framework of the D-brane paradigm [18].

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